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(Fractional) intersection numbers, tadpoles and anomalies

Christian Römelsberger

Department of Physics and Astronomy and CIT-USC Center for Theoretical Physics
University of Southern California
Los Angeles, CA 90089-0484, U.S.A.
E-mail: `cromelsberger@perimeterinstitute.ca`

Abstract: We use the Witten index in the open string sector to determine tadpole charges of orientifold planes and D-branes. As specific examples we consider type I compactifications on Calabi Yau manifolds and noncompact orbifolds. The tadpole constraints suggest that the standard embedding is not a natural choice for the gauge bundle. Rather there should be a close connection of the gauge bundle and the spin bundle. In the case of a four fold, the standard embedding does not in general fulfill the tadpole conditions. We show that this agrees with the Green-Schwarz mechanism. In the case of noncompact orbifolds we are able to solve the tadpole constraints with a gauge bundle, which is related to the spin bundle. We compare these results to anomaly cancellation on the fixed plane of the orbifold. In the case of branes wrapping noncompact cycles, there are fractional intersection numbers and anomaly coefficients, which we explain in geometric terms.

Keywords: Intersecting branes models, D-branes, Brane Dynamics in Gauge Theories, Anomalies in Field and String Theories.

Contents

1. Introduction

There has been a lot of progress in understanding D-branes in type II sting theory compactified on Calabi-Yau manifolds, even away from the geometrical regime. A first step in this direction was to identify charges of D-branes in nongeometrical regimes with geometrical charges (see e.g. [1 − 5]). In subsequent work the dynamics of these D-branes was understood further, (see e.g. [6 − 12]). All these approaches are dealing with D-branes which are point particles with different charges in transverse space.

In this note we want to put forward some foundations for the use of the previously described methods, in the context of space filling branes in type I theory. One of the differences is that these cannot be inserted in arbitrary numbers, but they have to fulfill some constraints due to the inconsistent RR flux they can produce. The cancellation of such tadpoles has been considered in numerous papers (see e.g. [13 − 19]).

The intersection of two D-branes is characterized by a massless fermionic string stretching between the two branes. Similarly, in the case of a D-brane and an O-plane the intersection is characterized by a massless fermionic string stretching between the D-brane and its image. The chirality of these fermionic strings determines the sign of the intersection. This intersection product is actually the intersection product of quantum K-theory.

We use this observation to determine the nontorsion tadpole charges of D-branes and O-planes, by calculating the Witten index of open string interactions with various 'probe' branes in a low energy description of the branes. This leads to a very quick procedure to determine the tadpole charges.

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In geometric Calabi Yau compactifications as well as in orbifold compactifications the tadpole constraints suggest a natural connection between the gauge bundle and the Dirac spinor bundle on the compactification manifold. This is opposed to the usual solution of the tadpole constraints on a 3-fold in terms of the standard embedding. In the case of $\mathbb{C}^d/Z_N$ orbifolds we find an explicit expression for the gauge bundle in terms of the spin bundle.

For noncompact spaces there arises another surprise. The intersection numbers between D-branes wrapping noncompact cycles turn out to be fractional. This is due to the continuous spectrum of momentum states in the noncompact directions. Geometrically this can be understood as coming from torsion of these cycles at the boundary of the noncompact space.

In section 2 we derive the general expressions for intersection numbers between two D-branes and between a D-brane and an O-plane. In section 3 we apply these general results to geometric Calabi-Yau compactifications and find agreement with anomaly cancellation. Finally in section 4 we explore orbifold compactifications. We find fractional intersection numbers between noncompact cycles and explain these in terms of relative homology. We solve the tadpole constraints for some noncompact orbifolds and compare these results to anomaly cancellation on the orbifold fixed plane.

2. Intersection numbers and tadpole charges

To illustrate the connection between intersection numbers and charges, let us look at type IIB string theory compactified on a two torus T^2. In order to determine in which way a D-string is winding around the torus, it is enough to determine its intersection numbers with two known D-strings wrapping the two fundamental 1-cycles of the torus.

At each intersection point of two D-strings on the torus, there is a massless fermionic string located. The chirality of this string indicates the sign of the intersection. In the following, we will generalize this concept of intersection numbers to more complicated brane configurations and also to orientifold planes.

In order to do this we want to describe a general formalism to calculate the torsion free part of the quantum K-theory charge of an orientifold plane. This can be done using the Witten index in the open string sector to define an intersection number [20, 21, 1].

In the CFT, the orientifold plane is described by a crosscap state and D-branes are described by boundary states, the closed string states which are emitted. The RR charge of a D-brane/O-plane is given by the RR ground state part of the corresponding boundary/crosscap state. To determine the RR charge it is useful to define a nondegenerate intersection form between boundary/crosscap states.

Given two boundary states $|E\rangle\rangle$ and $|F\rangle\rangle$ the intersection is defined as the overlap of the RR ground state part of the two boundary states, with a $(-)^{F_R}$ inserted in order to make it topological:

$$\begin{aligned} I(E,F) &= \langle E, RR-gs|(-)^{F_R}|F, RR-gs\rangle = \\ &= \langle\langle E, RR|(-)^{F_R} e^{-2\pi t H^{(cl)}}|F, RR\rangle\rangle. \end{aligned} \tag{2.1}$$

This (topological) cylinder amplitude can also be calculated in the open string sector

$$I(E,F) = \text{tr}\ _{R_{E,F}}(-)^F e^{-\frac{\pi}{t}H^{(op)}_{E,F}}. \tag{2.2}$$

This is in general easier than the calculation in the closed string sector and can often be done in the low energy theory of the D-branes.

Similarly, the intersection number between a crosscap state $|C\rangle\rangle$ and a boundary state $|E\rangle\rangle$ is the Möbius amplitude

$$I(C,E) = \langle\langle C, RR|(-)^{F_R} e^{-2\pi t(L_0 - \frac{c}{24})} e^{-2\pi t(\tilde{L}_0 - \frac{c}{24})}|E, RR\rangle\rangle. \tag{2.3}$$

Doing a modular transformation to the open string sector gives

$$I(C,E) = \text{tr}\ _{R_{E^*,E}}\Omega(-)^F e^{-\frac{\pi}{4t}(L_0 - \frac{c}{24})}. \tag{2.4}$$

The world-sheet parity operator Ω has to satisfy $\Omega^2 = 1$ in this open string sector and the Hilbert space can be divided into positive and negative parity eigenspaces.

3. The geometric Calabi Yau compactification

3.1 Tadpole analysis

To show how the formalism that we explained above works, we repeat the calculation of [22, 23] for geometric Calabi Yau compactifications. In the geometric case D-branes can be thought of as Chan-Paton bundles on the $2d$ dimensional Calabi Yau space X. The open string Ramond ground states between two D-branes E and F can be described by harmonic sections of

$$\Delta \otimes E^* \otimes F, \tag{3.1}$$

where Δ is the Dirac spinor bundle over X. The fermion number operator $(-)^F$ acts as the chirality operator on the spinors. From this can see that the intersection number (2.2) is the index of the twisted Dirac operator:

$$I(E,F) = \int_X \text{ch}(E^* \otimes F)\hat{A}(R). \tag{3.2}$$

This is the K-theoretic intersection number [24].

The action of Ω on open strings exchanges Chan-Paton factors in the fundamental representation of the gauge group on the one end of the string with Chan-Paton factors in the antifundamental representation of the gauge group on the other end of the string, i.e. it acts on the ends of an open string by $E \mapsto E^*$. This means that the open string Ramond ground states in the Möbius amplitude are harmonic sections of $\Delta \otimes E \otimes E$. Ω acts on these states simply by transposition $\gamma \mapsto \gamma^t$. In this way $\Delta \otimes E \otimes E$ is divided into the two Eigenspaces of Ω with Eigenvalues ± 1

$$\Delta \otimes E \otimes E = \Delta \otimes S^2 E \oplus \Delta \otimes \Lambda^2 E. \tag{3.3}$$

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The Möbius amplitude is now

$$Z_M = \int_X \mathrm{ch}(S^2E)\hat{A}(R) - \int_X \mathrm{ch}(\Lambda^2E)\hat{A}(R). \tag{3.4}$$

To fully calculate this expression we have to relate the Chern characters in the symmetric and antisymmetric representation to the Chern Characters of the fundamental representation

$$\begin{aligned}\mathrm{ch}(S^2E) &= \frac{1}{2}(\mathrm{ch}^2(E) + \mathrm{ch}(2E)),\\ \mathrm{ch}(\Lambda^2E) &= \frac{1}{2}(\mathrm{ch}^2(E) - \mathrm{ch}(2E)),\end{aligned} \tag{3.5}$$

where $\mathrm{ch}(2E)$ means that the curvatures in the expression for the Chern character are multiplied by 2. This gives the Möbius amplitude

$$Z_M = \int_X \mathrm{ch}(2E)\hat{A}(R) = 2^d \int_X \mathrm{ch}(E)\hat{A}\left(\frac{R}{2}\right). \tag{3.6}$$

Using trigonometric theorems the Möbius amplitude can be expressed as

$$Z_M = 2^d \int_X \mathrm{ch}(E)\sqrt{\hat{A}(R)}\sqrt{\hat{L}\left(\frac{R}{4}\right)}, \tag{3.7}$$

which shows that the Mukai charge of the crosscap state is $2^d\sqrt{\hat{L}\left(\frac{R}{4}\right)}$. For the full string theory 2^d can actually be replaced with $2^d \times 2^{5-d} = 32$ where the 2^{5-d} comes from the transverse dimensions.

In order to cancel the tadpoles one has to introduce a boundary state with a $\mathrm{Spin}(32)/Z_2$ gauge bundle E satisfying $\mathrm{ch}(E)\sqrt{\hat{A}(R)} = 32\sqrt{\hat{L}\left(\frac{R}{4}\right)}$, i.e.

$$\begin{aligned}\mathrm{p}_1(E) &= \mathrm{p}_1(T),\\ \mathrm{p}_2(E) &= -\frac{1}{8}\mathrm{p}_2(T) + \frac{15}{32}\mathrm{p}_1^2(T)\end{aligned} \tag{3.8}$$

in cohomology. These conditions might be trivially satisfied if X has low enough dimension. For example, the 8-form condition only applies on a 4-fold.

There is another interesting way to look at these conditions. Using the splitting principle the Chern character of the bundle E can be expressed as

$$\mathrm{ch}(E) = 2^d \prod_j \cosh\frac{x_j}{4}, \tag{3.9}$$

where the Dirac spinor bundle of X has the Chern character

$$\mathrm{ch}(\Delta) = 2^d \prod_j \cosh\frac{x_j}{2}. \tag{3.10}$$

This suggests, that the natural way to build the gauge bundle E is actually related to the spin bundle Δ and not via the standard embedding, which in general does not fulfill tadpole cancellation on a 4-fold. We will see a very similar condition later in the context of noncompact orbifolds. There we will be able to find an explicit solution to the analogous condition.

3.2 Comparison to the Green-Schwarz mechanism

The result from tadpole cancellation looks a little bit surprising from the point of view of the Green-Schwarz mechanism [25]. But as we will see in this section there arises the same 8-form condition from the Green-Schwarz mechanism. In the 10 dimensional type I supergravity there are three contributions to the chiral anomaly, the gravitino, the dilatino and the gauginos. Apart from these chiral fermions, there is a 2-form field B with a 3-form field strength H, which is used to cancel the chiral anomaly. The anomaly can be canceled, if the anomaly polynomial factorizes into a 4-form and a 8-form part

$$\hat{I}_{12} = X_4 \wedge X_8. \tag{3.11}$$

giving rise to the Bianchi identity

$$dH = X_4 \tag{3.12}$$

and the equation of motion

$$d * H = X_8. \tag{3.13}$$

These two equations imply that in the absence of 1- and 5-branes, the integral of X_4 and X_8 around any compact cycle has to vanish. It is not surprising that this agrees with the 4-form and 8-form conditions (3.8) from tadpole cancellation (see also equation. (3.46) in [26]).

In the context of heterotic strings it looks like the standard embedding should always work. This is only a statement in the perturbative nonlinear sigma model description. There should be inconsistencies appearing in the presence of NS5-branes [19, 27]. NS5-branes are the magnetic duals to the F-string and the 8-form condition is a condition on spacefilling F-strings.

4. Noncompact orbifolds

Let us now apply the same ideas to noncompact orbifolds $\mathbb{C}^d/\Gamma$ with isolated singularities only. The gauge theory part of such string theories can be described in terms of quiver diagrams [28, 29].

The new feature here is that we are dealing with space filling D($2d$)-branes, which are not localized on the orbifold singularities and by that token are not described by a four dimensional effective theory. The orbifold group has an action on $\mathbb{C}^d$ as well as on the Chan-Paton factors. The different irreducible representations on the Chan-Paton factors are as usual denoted by vertices of a quiver diagram. The arrows on the other hand behave slightly different than in the case of localized fractional branes. This is due to the fact that the open strings stretching between two such branes are not localized at the singularity and can propagate in the noncompact directions. This momentum part has nontrivial transformation properties under the orbifold group and can make up for some nontrivial transformation properties of the zero mode part.

In addition to this outer quiver, there is, of course, the well known inner quiver, describing fractional D0-branes. The fractional D0-branes represent branes wrapped around compact cycles [28, 30].

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4.1 Fractional intersection numbers in $\mathbb{C}^d/\Gamma$

To calculate the number of arrows between two fractional D($2d$)-branes one can make use of the character valued index theorem [31, 32]. For simplicity let us take a $\mathbb{C}^d/Z_N$ orbifold, where Z_N acts on $\mathbb{C}^d$ in a diagonal way $(z_1, \dots, z_d) \mapsto (\epsilon_1 z_1, \dots, \epsilon_d z_d)$, where $\epsilon_j^N = 1$ and $\prod \epsilon_j = 1$. For each irreducible representation of γ there exists a type of fractional D($2d$)-brane, which has this irreducible representation acting on its Chan-Paton factors. In the case of a Z_N orbifold these are the multiplication with N different phases.

To calculate the intersection numbers between different fractional branes, it is sufficient to keep only the massless open fermionic strings which propagate in $\mathbb{C}^d/\Gamma$. They are characterized by their Chan-Paton factors μ and ν, spinor degrees of freedom and a momentum. The Witten index (2.4) is now a trace over these massless fermionic strings with the chirality operator Γ^{2d+1} inserted. In flat space this trace vanishes, because the trace over the spinors vanishes, and it is surprising that in the untwisted sector of an orbifold theory this is not true. This comes about because the orbifold group action, which has to be inserted into the trace contains gamma matrices.

The index consists of three different traces, the trace over the spinors, the trace over the Chan-Paton factors and the integral over moment in the noncompact directions. For each complex direction, the action of the orbifold group on the spinors is given by

$$\cos\frac{\alpha_j}{2} + \sin\frac{\alpha_j}{2}\Gamma_{2j}\Gamma_{2j+1}, \tag{4.1}$$

where $\epsilon_j = e^{i\alpha_j}$. In order that the trace over the spinors is nonvanishing the second term in (4.1) has to be picked up. This gives a contribution of $(-i)^d 2^d \prod \sin\frac{\alpha_j}{2}$ from the trace over the spinors. The next contribution comes from the momentum integral

$$\int dp^{2d}\delta^{(2d)}(gp-p) = \frac{1}{|\det(1-g)|} = \frac{1}{4^d \prod\limits_j \sin^2\frac{\alpha_j}{2}}. \tag{4.2}$$

Finally there is a contribution $e^{2\pi i \frac{\mu-\nu}{N}}$ from the trace over the Chan-Paton factors. Putting all this together and summing over the orbifold group gives:

$$I^{(o)}_{\mu\nu} = \frac{(-i)^d}{N}\sum_m{}' \frac{e^{2\pi i\frac{\mu-\nu}{N}}}{2^d\prod\limits_j \sin\frac{\alpha_j m}{2}}, \tag{4.3}$$

where the prime indicates that the sum is only over terms which have a nonvanishing denominator, i.e. terms where the trace over the spinors does not vanish.

This expression for the intersection numbers is hard to simplify, but it will turn out in the following that they are fractional. The nonintegrality of the index itself is not inconsistent, because there is no energy gap between the ground states and states with momentum in the noncompact orbifold directions [33]. In the cases where the noncompact orbifold $\mathbb{C}^d/\Gamma$ can be embedded into a compact orbifold T^{2d}/Γ, the intersection numbers between D($2d$)-branes on the noncompact orbifold can be calculated by dividing the (integer) intersection number on the compact orbifold by the number of fixed points. These results agree with (4.3).

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In order to understand these fractional intersection numbers better, it is useful to calculate the intersection numbers of the inner quiver. They can be calculated in a similar manner. The only difference is the absence of the momentum integral:

$$I^{(i)}_{\mu\nu} = \frac{(-i)^d}{N} \sum_m e^{2\pi i \frac{\mu-\nu}{N}} 2^d \prod_j \sin \frac{\alpha_j m}{2}. \tag{4.4}$$

By expanding the sin's in terms of exponential functions it is easy to see that this is the same result as from counting the invariant chiral fields in the quiver gauge theory (see e.g. [28, 29]).

The intersection forms $I^{(i)}$ and $I^{(o)}$ are both degenerate and have a null vector corresponding to the pure D0-brane and the pure D$(2d)$-brane (regular representation). Taking the quotient of the two intersection 'lattices' by the null vectors, it is easy to see that the two lattices are, up to a sign of $(-)^d$, inverse to each other. This explains the fractionality of $I^{(o)}$.

The full intersection matrix of both, the inner and the outer quiver is

$$I = \begin{pmatrix} I^{(o)} & 1 \\ (-)^d & I^{(i)} \end{pmatrix}. \tag{4.5}$$

The rank of I is $N+1$. This shows, that there are only $N+1$ independent charges, either the pure D0-charge and N D$(2d)$-charges or the other way around.

4.2 Geometric explanation

The fractionality of the intersection numbers seems from the geometric point of view a little bit surprising, but it can be understood quite naturally in terms of relative homology. To illustrate the basic idea, it is useful to consider the example of $\mathbb{C}^2/Z_2$. The blow up of this orbifold is the total space of the line bundle $\mathcal{O}(-2) \xrightarrow{\pi} \mathbb{P}^1$.

There are two compact cycles, the point and the $\mathbb{P}^1$. It is easy to see, that the point doesn't intersect with any other compact cycle, but the $\mathbb{P}^1$ has a self intersection -2. This can be seen from the zeroes of a section of the normal bundle $\mathcal{O}(-2)$. The intersection matrix for the compact cycles is then

$$I^{(c)} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}. \tag{4.6}$$

The fractional D$(2d)$-branes are described by Chan-Paton bundles over the noncompact space $\mathcal{O}(-2)$. The bundles of interest are pull backs of line bundles over the base $\mathbb{P}^1$

$$\begin{gathered} \pi^* \mathcal{O}_{\mathbb{P}^1}, \\ \pi^* \mathcal{O}_{\mathbb{P}^1}(1). \end{gathered} \tag{4.7}$$

These branes have lower charges, which are the pull back of a point onto the fiber, i.e. the fiber itself. It is useful to keep track of the behavior at infinity of such a noncompact 2-cycle. The boundary at infinity of $\mathcal{O}(-2)$ is $S^3/Z_2 = \mathbb{R}P^3$ and the boundary of a fiber of

$\mathcal{O}(-2)$ is a noncontractible torsion 1-cycle in $\mathbb{R}P^3$. A 2-cycle which is wrapping the fiber twice has a trivial boundary in $\mathbb{R}P^3$ and can be contracted to a compact 2-cycle, a $\mathbb{P}^1$.

From the argument above it is easy to see, that the intersection number of two noncompact 2-cycles wrapping the fiber of $\mathcal{O}(-2)$ is $\frac{-2}{2\cdot 2} = -\frac{1}{2}$. This is the inverse of the nonzero eigenvalue of $I^{(c)}$.

We can now generalize this argument. Let X be a noncompact Calabi-Yau manifold and let $\dot{X}$ be it's boundary at infinity. Then compact cycles are described by the ordinary homology $H_*(X,Z)$. In order to describe noncompact cycles, it is useful to keep track of the behavior at infinity. This is done by the relative homology $H_*(X,\dot{X},Z)$.

An element of $H_*(X,\dot{X},Z)$ is denoted by the equivalence class $[\Gamma,\gamma]$, where $\Gamma \subset X$ and $\gamma \subset \dot{X}$. The relative boundary operator acts on a chain as

$$(\Gamma,\gamma) \mapsto \partial(\Gamma,\gamma) = (\partial\Gamma - \gamma, -\partial\gamma). \tag{4.8}$$

This can be understood as subtracting the boundary of Γ inside $\dot{X}$ from the regular boundary of Γ. The condition for a cycle to be closed, implies that $\partial\Gamma = \gamma$ and $\partial\gamma = 0$ as expected. The equivalence relation then becomes

$$[\Gamma,\gamma] \sim [\Gamma + \partial\Lambda - \lambda, \gamma - \partial\lambda]. \tag{4.9}$$

It is easy to see that Λ is the usual homology equivalence and λ is some piece in the boundary $\dot{X}$ that can be added.

It is easy to see that there is an exact sequence

$$\begin{array}{ccccc} H_p(X,Z) & \xrightarrow{i} & H_p(X,\dot{X},Z) & \xrightarrow{r} & H_{p-1}(\dot{X},Z) \\ [\Gamma] & \mapsto & [\Gamma,0] & & \\ & & [\Gamma,\gamma] & \mapsto & [\gamma] \end{array} \tag{4.10}$$

Any cycle $[\Gamma,\gamma]$ which restricts to torsion on the boundary, can be multiplied by the order N of the torsion[1]. $[N\Gamma, N\gamma]$ is then an element in $H_*(X,Z)$ and intersection numbers are well defined. Fractions are produced due to the multiplication by N.

Since cycles in $H_*(X,Z)$ are only in the interior of X, there is also a natural, integral intersection product between elements of $H_*(X,Z)$ and $H_*(X,\dot{X},Z)$. This intersection product is nondegenerate [34]. The intersection lattice $H_*(X,Z)$ modulo the null vectors is the dual to the intersection lattice of elements in $H_*(X,\dot{X},Z)$ which restrict to torsion on the boundary. There is an analogous statement in cohomology [35, 36], which is equivalent by Poincare duality.

This explanation also works quite well for $\mathbb{C}^3/Z_3$, which has S^3/Z_3 as boundary. The homology on S^3/Z_3 has been worked out [37], it has $H_1(S^3/Z_3,Z) = Z_3$ and $H_3(S^3/Z_3,Z) = Z_3$. The resolved orbifold is the total space of the bundle $\mathcal{O}(-3) \xrightarrow{\pi} \mathbb{P}^2$ and the arguments are very similar to the ones in $\mathbb{C}^2/Z_2$.

[1]Note that even though $[\gamma]$ is a torsion element in $H_*(\dot{X},Z)$, $[\Gamma,\gamma]$ is not necessarily a torsion element in $H_*(X,\dot{X},Z)$.

4.3 Solution to the tadpole constraints

In order to measure the charge of a crosscap state one can either calculate it's intersection with D0-probes or D($2d$)-probes. For comparison with anomaly cancellation on the fractional D9-branes it is useful to consider D($2d$)-probes.

In order to calculate the intersection product of a probe brane with a crosscap state we first have to determine the action of Ω on the R ground states. The action can be divided into three parts, the trivial action on the momentum, the action on the Chan-Paton factors and the action on the spinor indices.

The action on the fermion zero modes is $\psi_0^m \mapsto \pm\psi_0^m$, depending on whether there are N-N or D-D boundary conditions in the m-th direction. If there is an even number of D-D directions, then Ω acts as a chirality operator in these directions [14]. For the case of D($2d$)-branes this means that the Ω action on the spinors is trivial.

The requirement that the intersection of a D($2d$)-brane probe μ with the crosscap state is the same as its intersection with the tadpole canceling D($2d$)-brane boundary state can be summarized as

$$I^{(o)}_{\mu\Omega(\mu)} = \sum_\nu w_\nu I^{(o)}_{\nu\mu}, \tag{4.11}$$

together with the requirement of having a total of 2^d fractional D($2d$)-branes. Because we omitted the $10-2d$ transverse dimensions, there is a factor of 2^{5-d} missing on the right hand side of the equation. The final result has to be multiplied by $2^{(5-d)}$.

One would expect that for a high enough rank of the orbifold group, the equations (4.11) might not have a solution with nonnegative integer numbers w_ν of fractional D($2d$)-branes, leading to inconsistent backgrounds for type I theory. In the examples we consider, this actually turns out not to be the case.

A solution to these equations is given by a 2^d dimensional Chan-Paton representation of the orbifold group. The decomposition of this representation into irreps specifies the multiplicities of fractional D($2d$)-branes. The dimension of such a representation suggests a close connection to (Dirac) spinor representations.

Indeed for the $\mathbb{C}^d/Z_n$ orbifolds this guess turns out to be true. Equation (4.11) can be written as

$$\frac{(-i)^d}{N}\sum_m{}' e^{2\pi i\frac{2\mu}{N}m}\frac{1}{2^d\prod_j \sin\frac{\alpha_j m}{2}} = \sum_\nu w_\nu \frac{(-i)^d}{N}\sum_m{}' e^{2\pi i\frac{\mu-\nu}{N}m}\frac{1}{2^d\prod_j \sin\frac{\alpha_j m}{2}} \tag{4.12}$$

Using that N is odd and discrete Fourier transformation, this expression can be converted to

$$\sum_\nu w_\nu e^{2\pi i\frac{2\nu}{N}m} = 2^d\prod_j \cos\frac{\alpha_j m}{2}. \tag{4.13}$$

The left hand side of this equation is a sum over the characters of all irreps 2ν of Z_N with multiplicities w_ν. The right hand side is the character of the Dirac spinor representation associated to the geometrical action of Z_N. This shows that the Chan-Paton representation is almost the Spinor representation, except for a 'reshuffling' of the characters on the left hand side. This situation is similar to what we have seen in the geometric case in section 3.1.

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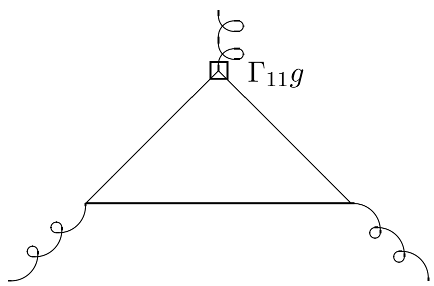

Figure 1: The anomalous diagram for an orbifold fixed plane.

4.4 Local anomalies in orbifolds

In order to check anomaly cancellation in orbifold theories, we need to derive an expression for the anomaly on a fixed plane of an orbifold due to the untwisted fields of the theory. An anomaly on the fixed plane due to fields from the twisted sector can be calculated in a straight forward way by the descent formalism. The derivation in this section is very similar to the derivation of the index in section 4.1, but we want to do this derivation in a bit more detail, because it is also quite important for a more elementary understanding of anomalies in orbifolds of M-theory [38 – 40].

Typically the one loop chiral anomaly due to ten dimensional fields is calculated with the help of a hexagon diagram [41], giving rise to a ten dimensional anomaly. The anomaly on a $(10-2d)$-dimensional orbifold fixed plane has to be derived from a $(6-d)$-polygon diagram with the ten dimensional chiral fields running around in the loop. This can be nonvanishing because of the insertion of the gamma matrices from the orbifold group action g in the loop (see figure 1). The momentum integral is still over a ten dimensional momentum.

The momenta and polarizations of the external lines can be set in the direction of the orbifold fixed plane, then the traces in the diagram can be split into traces inside the fixed plane and traces transverse to the fixed plane. The traces transverse to the fixed plane are actually the same traces that lead to the index (4.3). The traces inside the orbifold fixed plane are exactly the same as for the chiral anomaly of a $(10-2d)$-dimensional fermion. So we conclude that the chiral anomaly on the fixed plane from the ten dimensional fermion fields is the chiral anomaly of a $(10-2d)$-dimensional fermion with the index (4.3) as a prefactor.

To check this result and to understand the fractionality of the prefactor (4.3), we want to look at some elementary examples. These are all orbifolds of tori. The T^6/Z_3 for example has 27 fixed planes. The anomaly on a single fixed plane is equal to the four dimensional chiral anomaly of the dimensionally reduced theory. This four dimensional anomaly is equally distributed over all 27 fixed planes. In the dimensionally reduced theory there are three chiral fermions between two different fractional D9-branes, which leads to a prefactor of $\frac{3}{27}=\frac{1}{9}$ for the chiral anomaly of each fixed plane. This agrees with the result from (4.3). One can look at the orbifolds T^4/Z_2 and T^4/Z_3 in a similar way.

The chiral anomaly of the gravitino can be calculated in a similar way. There are two different contributions from the gravitino, one where the vector index of the gravitino is in the orbifold fixed plane, it is invariant under the orbifold group, this gives rise to a $(10-2d)$-dimensional gravitino anomaly with a prefactor (4.3). If the vector index is transverse to the orbifold fixed plane, the orbifold group acts on it with the regular representation. This case is treated in the same way as spin $\frac{1}{2}$ fermions.

A ten dimensional gravitino gives in this way rise to $(10-2d)$-dimensional 'gravitino' and 'dilatino' anomalies. In the case of $d=3$ it is easy to see that the four dimensional anomalies due to the ten dimensional gravitino vanish, because the sum (4.3) is over an odd function.

To conclude this section we want to make a quick comment on chiral anomalies in orbifolds of M-theory [38 – 40]. For M-theory on $\mathbb{R}/Z_2$ the relevant diagram is the hexagon diagram. The orbifold group acts on spinors with a Γ_{11}, and a prefactor of $\frac{1}{2}$ comes from the rank of the orbifold group. This shows that on each fixed plane in the Horava-Witten picture there is half of the ten dimensional anomaly. A similar argument applies for M-theory on T^5/Z_2.

4.5 Anomalies in quiver theories

Now we want to investigate the anomaly cancellation in the case of noncompact orbifolds. In the case of compact orbifolds there have been quite detailed studies (see e.g. [42]). The anomaly conditions in the case of quiver theories are a bit weaker than in the ten dimensional case. We consider a theory which is compactified on an orbifold down to four dimensions.

Away from the orbifold singularities, anomaly cancellation implies the same as for smooth compactifications. The anomaly polynomial on a four dimensional fixed plane is a 6-form polynomial of the form

$$\hat{I}_6 = I[\mathrm{ch}(E_{\mathrm{tot}})\hat{A}(R)]_6, \tag{4.14}$$

I being the prefactor (4.3). Each term in this polynomial has to factorize into 2-form and 4-form parts. The term which potentially might not factorize this way is tr F^3 for any factor of the gauge group. Only unitary factors of the gauge group actually have a nonvanishing tr F^3. If such a $\mathrm{U}(N)$ factor has $N>2$ then this trace (in the fundamental representation) does not factorize and creates an anomaly which cannot be canceled by a closed string twist field living on the fixed plane.

In the orientifolded theory such $\mathrm{U}(N)$ factors arise from vertices which are not fixed under the Z_2 involution Ω. Such a vertex μ can have two different kinds of arrows ending on it, arrows which are fixed under the involution and arrows which are not fixed. The latter ones give rise to matter in the fundamental or anti-fundamental representation of the gauge group $\mathrm{U}(w_\mu)$ depending on the direction of the arrow.

The fixed arrows give rise to antisymmetric representations. The Chern characters in the antisymmetric representations can be expressed in terms of Chern characters in the fundamental or antifundamental representation of $\mathrm{U}(w_\mu)$ depending again on the direction of the arrow.

$$\begin{aligned} \mathrm{ch}_3(\Lambda^2 E) &= r(E)\mathrm{ch}_3(E) - 4\mathrm{ch}_3(E) + \cdots , \\ \mathrm{ch}_3(\Lambda^2 E^*) &= -r(E)\mathrm{ch}_3(E) + 4\mathrm{ch}_3(E) + \cdots , \end{aligned} \tag{4.15}$$

where the dots indicate terms which don't involve $\mathrm{ch}_3(E)$.

The constant in front of $\mathrm{ch}_3(E)$ in the anomaly polynomial can be written as

$$\sum_{\nu\neq\Omega(\mu)} w_\nu I^{(o)}_{\nu\mu} + w_\mu I^{(o)}_{\mu\Omega(\mu)} - 4I^{(o)}_{\mu\Omega(\mu)} = \sum_\nu w_\nu I^{(o)}_{\nu\mu} - 4I^{(o)}_{\mu\Omega(\mu)}. \tag{4.16}$$

The vanishing of (4.16) follows from tadpole cancellation (4.11), but the tadpole condition is stronger, because there are also conditions for fixed vertices and $\mathrm{U}(N \leq 2)$ gauge groups.

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References

[1] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Römelsberger, *D-branes on the quintic*, *JHEP* **08** (2000) 015 [hep-th/9906200].

[2] D. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *JHEP* **10** (2000) 001.

[3] D.E. Diaconescu and C. Römelsberger, *D-branes and bundles on elliptic fibrations*, *Nucl. Phys.* **B 574** (2000) 245 [hep-th/9910172].

[4] P. Kaste, W. Lerche, C.A. Lutken and J. Walcher, *D-branes on K3-fibrations*, *Nucl. Phys.* **B 582** (2000) 203 [hep-th/9912147].

[5] E. Scheidegger, *D-branes on some one- and two-parameter Calabi-Yau hypersurfaces*, *JHEP* **04** (2000) 003 [hep-th/9912188].

[6] M.R. Douglas, B. Fiol and C. Römelsberger, *Stability and BPS branes*, *JHEP* **09** (2005) 006 [hep-th/0002037].

[7] M.R. Douglas, B. Fiol and C. Römelsberger, *The spectrum of BPS branes on a noncompact Calabi-Yau*, *JHEP* **09** (2005) 057 [hep-th/0003263].

[8] D.E. Diaconescu and M.R. Douglas, *D-branes on stringy Calabi-Yau manifolds*, hep-th/0006224.

[9] P. Mayr, *Phases of supersymmetric D-branes on kaehler manifolds and the McKay correspondence*, *JHEP* **01** (2001) 018 [hep-th/0010223].

[10] S. Govindarajan and T. Jayaraman, *D-branes, exceptional sheaves and quivers on Calabi-Yau manifolds: from Mukai to McKay*, *Nucl. Phys.* **B 600** (2001) 457 [hep-th/0010196].

[11] A. Tomasiello, *D-branes on Calabi-Yau manifolds and helices*, *JHEP* **02** (2001) 008 [hep-th/0010217].

[12] M.R. Douglas, *D-branes and $N = 1$ supersymmetry*, hep-th/0105014.

[13] M.B. Green and J.H. Schwarz, *Infinity cancellations in SO(32) superstring theory.*

[14] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[15] E.G. Gimon and C.V. Johnson, *K3 orientifolds*, *Nucl. Phys.* **B 477** (1996) 715 [hep-th/9604129].

[16] J.D. Blum and K.A. Intriligator, *Consistency conditions for branes at orbifold singularities*, *Nucl. Phys.* **B 506** (1997) 223 [hep-th/9705030].

[17] G. Aldazabal, D. Badagnani, L.E. Ibáñez and A.M. Uranga, *Tadpole versus anomaly cancellation in $D = 4$, 6 compact IIB orientifolds*, *JHEP* **06** (1999) 031 [hep-th/9904071].

[18] G. Aldazabal, A. Font, L.E. Ibáñez and G. Violero, *$D = 4$, $N = 1$, type-IIB orientifolds*, *Nucl. Phys.* **B 536** (1998) 29 [hep-th/9804026].

[19] A.M. Uranga, *D-brane probes, RR tadpole cancellation and K-theory charge*, *Nucl. Phys.* **B 598** (2001) 225 [hep-th/0011048].

[20] M. Berkooz and M.R. Douglas, *Five-branes in M(atrix) theory*, *Phys. Lett.* **B 395** (1997) 196 [hep-th/9610236].

[21] M.R. Douglas and B. Fiol, *D-branes and discrete torsion, II*, *JHEP* **09** (2005) 053 [hep-th/9903031].

[22] J.F. Morales, C.A. Scrucca and M. Serone, *Anomalous couplings for D-branes and O-planes*, *Nucl. Phys.* **B 552** (1999) 291 [hep-th/9812071].

[23] B.J. Stefanski, *Gravitational couplings of D-branes and O-planes*, *Nucl. Phys.* **B 548** (1999) 275 [hep-th/9812088].

[24] R. Minasian and G.W. Moore, *K-theory and Ramond-Ramond charge*, *JHEP* **11** (1997) 002 [hep-th/9710230].

[25] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory*, *Phys. Lett.* **B 149** (1984) 117.

[26] D.S. Freed, *Dirac charge quantization and generalized differential cohomology*, hep-th/0011220.

[27] M.J. Duff and J.X. Lu, *Elementary five-brane solutions of $D = 10$ supergravity*, *Nucl. Phys.* **B 354** (1991) 141.

[28] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[29] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[30] D.E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *JHEP* **02** (1998) 013 [hep-th/9712230].

[31] M.W. Goodman, *Proof of character valued index theorems*, *Commun. Math. Phys.* **107** (1986) 391.

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[32] T. Eguchi, P.B. Gilkey and A.J. Hanson.

[33] A.J. Niemi and G.W. Semenoff, *Fermion number fractionization in quantum field theory*, *Phys. Rept.* **135** (1986) 99.

[34] E.H. Spanier, *Algebraic Topology*, Mc Graw-Hill, 1966.

[35] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four-folds*, *Nucl. Phys.* **B 584** (2000) 69 [hep-th/9906070].

[36] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Springer-Verlag, 1982.

[37] S. Gukov, M. Rangamani and E. Witten, *Dibaryons, strings and branes in AdS orbifold models*, *JHEP* **12** (1998) 025 [hep-th/9811048].

[38] P. Hořava and E. Witten, *Heterotic and type-I string dynamics from eleven dimensions*, *Nucl. Phys.* **B 460** (1996) 506 [hep-th/9510209].

[39] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 094.

[40] E. Witten, *Five-branes and M-theory on an orbifold*, *Nucl. Phys.* **B 463** (1996) 383 [hep-th/9512219].

[41] L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys.* **B 234** (1984) 269.

[42] M. Berkooz et al., *Anomalies, dualities and topology of* $D = 6$ $N = 1$ *superstring vacua*, *Nucl. Phys.* **B 475** (1996) 115 [hep-th/9605184].